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ON STRONG M_α-INTEGRAL OF BANACH-VALUED FUNCTIONS

Xuexiao You*, Jian Cheng**, and Dafang Zhao***

Abstract. In this paper, we define the Banach-valued strong M_α-integral and study the primitive of the strong M_α-integral in terms of the M_α-variational measures. We also prove that every function of bounded variation is a multiplier for the strong M_α-integral.

1. Introduction

In [1], Jae Myung Park, Hyung Won Ryu and Hoe Kyoung Lee introduced a Riemann type integration process, called M_α-integral, which falls in between the Lebesgue Integral and the Henstock Integral. Some properties of the M_α-integral were studied in $[1, 2, 3]$.

In this paper, we define and study the strong M_α-integral of functions mapping an interval [a,b] into a Banach space X. We prove that the M_α-integral and the strong M_α-integral are equivalent if and only if the Banach space is finite dimensional. If the function $F : \mathcal{I} \to X$ is differentiable almost everywhere on $[a, b]$, then it is the indefinite strong M_α-integral of f if and only if the M_α-variational measure V_*F is absolutely continuous. Consequently, we prove that every function of bounded variation is a multiplier for the strong M_α-integral.

2. Definitions and basic properties

Throughout this paper, α is a positive real number, $[a, b]$ is a compact interval in R. X will denote a real Banach space with norm $\|\cdot\|$ and its dual X^*. $\mathcal{I}$ denote the family of all subintervals of $[a, b]$. $\overline{co}(Y)$ denote the closed convex hull of the set Y if $Y \subset X$.

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Correspondence should be addressed to Dafang Zhao, dafangzhao@163.com.

A partition D is a finite collection of interval-point pairs $\{([u_i, v_i], \xi_i)\}_{i=1}^n$, where $\{[u_i, v_i]\}_{i=1}^n$ are non-overlapping subintervals of $[a, b]$. $\delta(\xi)$ is a positive function on $[a, b]$, i.e. $\delta(\xi) : [a, b] \to \mathrm{R}^+$. We say that $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is

(1) a partial partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] \subset [a, b]$,
(2) a partition of $[a, b]$ if $\bigcup_{i=1}^n [u_i, v_i] = [a, b]$,
(3) δ-fine *McShane partition* of $[a, b]$ if $[u_i, v_i] \subset B(\xi_i, \delta(\xi_i))$ $= (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in [a, b]$ for all i=1, 2, $\cdots$, n,
(4) δ-fine M_α*-partition* of $[a, b]$ if it is a δ-fine McShane partition of $[a, b]$ and satisfying the condition

$$\sum_{i=1}^n dist(\xi_i, [u_i, v_i]) < \alpha$$

for the given α, here $dist(\xi_i, [u_i, v_i]) = inf\{|t_i - \xi_i| : t_i \in [u_i, v_i]\}$.

Given a δ-fine M_α*-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ we write

$$S(f, D) = \sum_{i=1}^n f(\xi_i)(v_i - u_i)$$

for integral sums over D, whenever $f : [a, b] \to X$.

Definition 2.1. A function $f : [a, b] \to X$ is M_α-integrable if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that

$$\|S(f, D) - A\| < \epsilon$$

for each δ-fine M_α*-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$. A is called the M_α*-integral* of f on $[a, b]$, and we write $A = \int_a^b f$ or $A = (M_\alpha) \int_a^b f$.

The function f is M_α-integrable on the set $E \subset [a, b]$ if the function $f\chi_E$ is M_α-integrable on $[a, b]$. We write$\int_E f = \int_a^b f\chi_E$.

The basic properties of the M_α-integral, for example, linearity and additivity with respect to intervals can be founded in [3]. We do not present them here. The reader is referred to [3] for the details.

Lemma 2.2. *(Saks-Henstock) Let $f : [a, b] \to X$ is M_α-integrable on $[a, b]$. Then for $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that*

$$\|S(f, D) - \int_a^b f\| < \epsilon$$

for each δ-fine M_α-partition $D = \{([u,v],\xi)\}$ of $[a,b]$. Particulary, if $D' = \{([u_i,v_i],\xi_i)\}_{i=1}^m$ is an arbitrary δ-fine *partial M_α-partition* of $[a,b]$, we have

$$\|S(f,D') - \sum_{i=1}^m \int_{u_i}^{v_i} f\| \le \epsilon.$$

Proof. The reader is referred to [3, Lemma 2.5] for the details. □

THEOREM 2.3. *Let $f : [a,b] \to X$ is M_α-integrable on $[a,b]$.*

(1) *for each $x^* \in X^*$, the function $x^* f$ is M_α-integrable on $[a,b]$ and $\int_a^b x^* f = x^*(\int_a^b f)$.*
(2) *If $T : X \to Y$ is a continuous linear operator, then Tf is M_α-integrable on $[a,b]$ and $\int_a^b Tf = T(\int_a^b f)$.*

Proof. The proof is too easy and will be omitted. □

DEFINITION 2.4. A function $f : [a,b] \to X$ is strongly M_α-integrable if there exists an additive function $F : \mathcal{I} \to X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a,b] \to R^+$ such that

$$\sum_{i=1}^n \|f(\xi_i)(v_i - u_i) - F(u_i,v_i)\| < \epsilon$$

for each δ - fine *M_α-partition* $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ of $[a,b]$. We denote $F(u_i,v_i) = F(v_i) - F(u_i)$.

THEOREM 2.5. *Let X be a Banach space of finite dimension. $f : [a,b] \to X$ is M_α-integrable on $[a,b]$ if and only if f is strongly M_α-integrable on $[a,b]$.*

Proof. **Sufficiency:** From the definitions of the strong M_α-integral and M_α-integral, if f is strongly M_α-integrable on $[a,b]$, then f is M_α-integrable on $[a,b]$.

Necessity: f is M_α-integrable on $[a,b]$, then there is a positive function $\delta(\xi) : [a,b] \to R^+$ such that

$$\Big\| \sum f(\xi)(v-u) - F(u,v) \Big\| < \epsilon$$

for each δ -fine *M_α-partition* $D = \{([u,v],\xi)\}$ of $[a,b]$. Let $\{e_1, e_2, \cdots, e_n\}$ be a base of X and $g_i : [a,b] \to R$ $(i = 1,2,\cdots,n)$. By the Hahn-Banach Theorem, for each e_i there is $x_i^* \in X^*$ such that

$$x_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases} \tag{2.1}$$

for $i, j = 1, 2, \cdots, n$ and therefore $x_i^*(f) = \sum_{j=1}^n g_j x_i^*(e_j) = g_i$. Since $g_i : [a,b] \to R$ is M_α-integrable on $[a,b]$ from Theorem 2.3, for each $\varepsilon > 0$ there is a positive function $\delta_i(\xi) : [a,b] \to R^+$ such that

$$|S(g_i, D_i) - \sum \int_u^v g_i| < \epsilon$$

for each δ_i - fine M_α*-partition* $D_i = \{([u,v], \xi)\}$ of $[a,b]$. By an easy adaptation of Saks-Henstock Lemma we have

$$\sum |g_i(\xi)(v-u) - \int_u^v g_i| < 2\epsilon.$$

We also have

$$F(u,v) = \int_u^v f = \int_u^v \sum_{i=1}^n g_i e_i = \sum_{i=1}^n \int_u^v g_i e_i = \sum_{i=1}^n e_i G_i(u,v)$$

where $G_i(u,v) = \int_u^v g_i$. Let $\delta(\xi) < \delta_i(\xi)$ for $i = 1, 2, \cdots, n$ and consequently

$$\begin{aligned} &\sum \big\| f(\xi)(v-u) - F(u,v) \big\| \\ &= \sum \big\| \sum_{i=1}^n g_i(\xi) e_i (v-u) - \sum_{i=1}^n e_i G_i(u,v) \big\| \\ &\le \sum_{i=1}^n \big\| e_i \big\| \sum |g_i(\xi)(v-u) - G_i(u,v)| \\ &< \epsilon \cdot \sum_{i=1}^n \| e_i \| \end{aligned}$$

for each δ-fine M_α*-partition* $D = \{([u,v], \xi)\}$ of $[a,b]$. Hence f is strongly M_α-integrable on $[a,b]$. □

3. The M_α-variational measure and the strong M_α-integral

Let $F : [a,b] \to X$, arbitrary $E \subset [a,b]$ and a positive function $\delta(\xi) : E \to R^+$, Let us set

$$V(F, \delta, E) = \sup_D \sum_i \|F(u_i, v_i)\|$$

where the supremum is take over all δ - fine *partial* M_α*-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a,b]$ with $\xi_i \in E$. We put

$$V_* F(E) = \inf_\delta V(F, \delta, E)$$

where the infimum is take over all function $\delta(\xi) : E \to R^+$.

It is easy to know that the set function $V_*F(E)$ is a Borel metric outer measure, known as the M_α-variational measure generated by F.

DEFINITION 3.1. $V_*F(E)$ is said to be absolutely continuous (AC) on a set E if for each set $N \subset E$ such that $V_*F(N) = 0$ whenever $\mu(N) = 0$.

DEFINITION 3.2. A function $F : [a, b] \to X$ is differentiable at $\xi \in [a, b]$ if there is a $f(\xi) \in X$ such that

$$\lim_{\delta \to 0} \left\| \frac{F(\xi + \delta) - F(\xi)}{\delta} - f(\xi) \right\| = 0.$$

We denote $f(\xi) = F'(\xi)$ the derivative of F at ξ.

THEOREM 3.3. *Let $F : \mathcal{I} \to X$ be differentiable almost everywhere on $[a, b]$. Then F is the indefinite strong M_α-integral of f if and only if the M_α-variational measure V_*F is AC.*

Proof. **Necessity:** Let $E \subset [a, b]$ and $\mu(E) = 0$. Assume $E_n = \{\xi \in E : n - 1 \leq \|f(\xi)\| < n\}$ for $n = 1, 2, \cdots$. Then we have $E = \bigcup E_n$ and $\mu(E_n) = 0$, so there are open sets G_n such that $E_n \subset G_n$ and $\mu(G_n) < \frac{\epsilon}{n \cdot 2^n}$.

By the Saks-Henstock Lemma, there exists a positive function δ_0 such that

$$\sum \|f(\xi_i)(v_i - u_i) - F(u_i, v_i)\| < \epsilon$$

for each δ_0 - fine *partial M_α-partition* $D = \{([u_i, v_i], \xi_i)\}$ of $[a, b]$.

For $\xi \in E_n$, take $\delta_n(\xi) > 0$ such that $B(\xi, \delta_n(\xi)) \subset G_n$. Let

$$\delta(\xi) = min\{\delta_0(\xi), \delta_n(\xi)\}.$$

Assume $D' = \{([u, v], \xi)\}$ is a δ - fine *partial M_α-partition* with $\xi \in E$. We have

$$\begin{aligned}
\sum \|F(u, v)\| &= \sum \|F(u, v) - f(\xi)(v - u) + f(\xi)(v - u)\| \\
&\leq \sum \|F(u, v) - f(\xi)(v - u)\| + \sum \|f(\xi)(v - u)\| \\
&< \epsilon + \sum_n \sum_{\xi \in E_n} \|f(\xi)(v - u)\| \\
&< \epsilon + \sum_n n \frac{\epsilon}{n \cdot 2^n} = 2\epsilon
\end{aligned}$$

This shows that $V_*F(E) < 2\epsilon$. Hence the M_α-variational measure V_*F is AC as desired.

Sufficiency: There exists a set $E \subset [a, b]$ be of measure zero such that $f(\xi) \neq F'(\xi)$ or $F'(\xi)$ does not exist for $\xi \in E$. We can define a function as follows

$$f(x) = \begin{cases} F'(\xi) & \text{if } \xi \in [a, b]\backslash E, \\ \theta & \text{if } \xi \in E. \end{cases} \tag{3.1}$$

Then for $\xi \in [a, b]\backslash E$, by the definition of derivative, for each $\varepsilon > 0$, there is a positive function $\delta_1(\xi)$ such that

$$\|f(\xi)(v-u) - F(u,v)\| < \frac{\epsilon}{\alpha + (b-a)}(dist(\xi, [u,v]) + v - u)$$

for each interval $[u, v] \subset (\xi - \delta_1(\xi), \xi + \delta_1(\xi))$.

V_*F is AC, then for $\xi \in E$, there is a positive function $\delta_2(\xi)$ such that

$$\sum \|F(u,v)\| < \epsilon$$

for each δ_2 - fine *partial* M_α*-partition* $D_0 = \{([u, v], \xi)\}$ with $\xi \in E$.

Define a positive function $\delta(\xi)$ as follows

$$\delta(\xi) = \begin{cases} \delta_1(\xi) & \text{if } \xi \in [a, b]\backslash E, \\ \delta_2(\xi) & \text{if } \xi \in E. \end{cases} \tag{3.2}$$

Then for each δ - fine M_α*-partition of* $[a, b]$, we have

$$\begin{aligned} &\sum \|f(\xi)(v-u) - F(u,v)\| \\ &= \sum_{\xi \in E} \|F(u,v) - f(\xi)(v-u)\| + \sum_{\xi \in [a,b]\backslash E} \|F(u,v) - f(\xi)(v-u)\| \\ &\leq \epsilon + \frac{\epsilon}{\alpha + (b-a)} \sum_{\xi \in [a,b]\backslash E} (dist(\xi, [u,v]) + v - u) \\ &< \epsilon + \frac{\epsilon}{\alpha + (b-a)}(\alpha + b - a) = 2\epsilon. \end{aligned}$$

Hence f is strong M_α-integrable on $[a, b]$ with indefinite strong M_α-integral F. □

DEFINITION 3.4. Let $G : [a, b] \to R$. A function $F : [a, b] \to X$ is M_α-Stieltjes integrable with respect to G on $[a, b]$ if there exists a vector $A \in X$ such that for each $\varepsilon > 0$ there is a positive function $\delta(\xi) : [a, b] \to R^+$ such that

$$\|S(F, G, D) - A\| < \epsilon$$

for each δ-fine M_α*-partition* $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of $[a, b]$, whenever

$$S(F, G, D) = \sum_{i=1}^{n} F(\xi_i)(G(v_i) - G(u_i)).$$

A is called the M_α *-Stieltjes integral* of F with respect to G on $[a, b]$, and we write $A = (M_\alpha S) \int_a^b F dG$.

Similar to [10, Proposition 5], we have the following Lemma.

LEMMA 3.5. *Let $G : [a, b] \to R$ be a non decreasing function. If a function $F : [a, b] \to X$ is M_α-Stieltjes integrable with respect to G, then for each $[u, v] \in [a, b]$, we have*

$$(M_\alpha S) \int_u^v F dG \in \overline{co}(\{G(u, v)x : x \in X \, and \, x = F(\xi) \, for \, some \, \xi \in [u, v]\}).$$

THEOREM 3.6. *Let $f : [a, b] \to X$ be strongly M_α-integrable on $[a, b]$ and $F(x) = \int_a^x f$ for each $x \in [a, b]$. If $G : [a, b] \to R$ is a function of bounded variation, then Gf is strongly M_α-integrable and*

$$\int_a^b Gf = G(b)F(b) - (M_\alpha S) \int_a^b F dG.$$

Proof. Let $\epsilon > 0$, arbitrary $E \subset [a, b]$ with $\mu(E) = 0$. Assume $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is an arbitrary δ - fine *partial* M_α*-partition* with $\xi_i \in E$.

It is easy to know that F is continuous on $[a, b]$. G is of bounded variation, then the M_α-Stieltjes integral $(M_\alpha S) \int_a^b F dG$ exists on $[a, b]$. We can assume G is non decreasing and with upper bounded $M > 0$ on $[a, b]$, then for each i, there are $x_1^{(i)}, x_2^{(i)}, \cdots, x_{m_i}^{(i)} \in [u_i, v_i]$ and numbers $\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{m_i}^{(i)}$ with $\sum_{j=1}^{m_i} \lambda_j^{(i)} = 1$ such that

$$\| \sum_{j=1}^{m_i} \lambda_j^{(i)} G(u_i, v_i) F(x_j^{(i)}) - \int_{u_i}^{v_i} F dG \| \leq \frac{\epsilon G(u_i, v_i)}{n V(G, [a, b])}$$

where $V(G, [a, b])$ denote the variation of G over the interval $[a, b]$.

We define a function by

$$\int_a^x Gf = H(x) = G(x)F(x) - (M_\alpha S) \int_a^x F dG$$

and consequently have

$$\begin{aligned}
&\|H(v_i)-H(u_i)\| \\
&=\|G(v_i)F(v_i)-G(u_i)F(u_i)-(M_\alpha S)\int_{u_i}^{v_i}FdG\| \\
&=\|G(v_i)[F(v_i)-F(u_i)]+(G(v_i)-G(u_i))[F(u_i)-\sum_{j=1}^{m_i}\lambda_j^{(i)}F(x_j^{(i)})] \\
&\quad +(G(v_i)-G(u_i))\sum_{j=1}^{m_i}\lambda_j^{(i)}F(x_j^{(i)})-(M_\alpha S)\int_{u_i}^{v_i}FdG\| \\
&\leq |G(v_i)|\cdot\|F(u_i,v_i)\|+G(u_i,v_i)\|F(u_i)-\sum_{j=1}^{m_i}\lambda_j^{(i)}F(x_j^{(i)})\| \\
&\quad +|G(u_i,v_i)\sum_{j=1}^{m_i}\lambda_j^{(i)}F(x_j^{(i)})-(M_\alpha S)\int_{u_i}^{v_i}FdG\| \\
&\leq M\|F(u_i,v_i)\|+G(u_i,v_i)\|\sum_{j=1}^{m_i}\lambda_j^{(i)}[F(u_i)-F(x_j^{(i)})]\|+\frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])} \\
&\leq M\|F(u_i,v_i)\|+V(G,[a,b])\sum_{j=1}^{m_i}\lambda_j^{(i)}\|F(u_i)-F(x_j^{(i)})\|+\frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])} \\
&\leq M\|F(u_i,v_i)\|+V(G,[a,b])\sum_{j=1}^{m_i}\lambda_j^{(i)}\|F(u_i)-F(x_{N^{(i)}}^{(i)})\|+\frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])} \\
&=M\|F(u_i,v_i)\|+V(G,[a,b])\|F(u_i)-F(x_{N^{(i)}}^{(i)})\|+\frac{\epsilon G(u_i,v_i)}{nV(G,[a,b])}
\end{aligned}$$

where $\|F(u_i)-F(x_{N^{(i)}}^{(i)})\|=max\{\|F(u_i)-F(x_j^{(i)})\|\}$for each $j\in\{1,2,\cdots,m_i^{(i)}\}$. V_*F is AC from Theorem 3.3, then there exists a positive function $\delta(\xi)$ such that

$$\sum_{i=1}^{n}\|F(u_i,v_i)\|<\frac{\epsilon}{M+V(G,[a,b])}.$$

Therefore

$$\begin{aligned}\sum_{i=1}^{n}\|H(v_i)-H(u_i)\| &\leq M\sum_{i=1}^{n}\|F(u_i,v_i)\|+\frac{\epsilon\sum_{i=1}^{n}G(u_i,v_i)}{nV(G,[a,b])}\\ &\quad +V(G,[a,b])\sum_{i=1}^{n}\|F(u_i)-F(x_{N(i)}^{(i)})\|\\ &\leq (M+V(G,[a,b]))\frac{\epsilon}{M+V(G,[a,b])}+\epsilon=2\epsilon\end{aligned}$$

and it follows that V_*H is AC. We also have that $H(x)$ is differentiable almost everywhere and $H'(x)=G(x)f(x)$ a.e. on $[a,b]$, then Gf is strongly M_α-integrable on $[a,b]$ from Theorem 3.3. □

Consequently, we can easily get the following theorem.

THEOREM 3.7. *Let $f:[a,b]\to X$ and $G:[a,b]\to R$. If Gf is strongly M_α-integrable on $[a,b]$ for every strongly M_α-integrable f, then G is equivalent to a function of bounded variation on $[a,b]$.*

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School of Mathematics and Statistics
Hubei Normal University
Huangshi 435002, China
E-mail: youxuexiao@126.com

**
School of Mathematics and Statistics
Hubei Normal University
Huangshi 435002, China
E-mail: chengjian@163.com

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School of Mathematics and Statistics
Hubei Normal University
Huangshi 435002, China
E-mail: dafangzhao@163.com